

# Bouncing Dirac particles: compatibility between MIT boundary conditions and Thomas precession

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## Abstract

We consider the reflection of a Dirac plane wave on a perfectly reflecting plane described by chiral MIT boundary conditions and determine the rotation of the spin in the reflected component of the wave. We solve the analogous problem for a classical particle using the evolution of the spin defined by the Thomas precession and make a comparison with the quantum result. We find that the rotation axes of the spin in the two problems coincide only for a vanishing chiral angle, in which case the rotation angles coincide in the nonrelativistic limit, and also remain remarkably close in the relativistic regime. The result shows that in the nonrelativistic limit the interaction between the spin and a reflecting surface with nonchiral boundary conditions is completely contained in the Thomas precession effect, in conformity with the fact that these boundary conditions are equivalent to an infinite repulsive scalar potential outside the boundary. By contrast, in the ultrarelativistic limit the rotation angle in the quantum problem remains finite, while in the classical one the rotation angle diverges. We comment on the possible implications of this discrepancy on the validity of the Mathisson-Papapetrou-Dixon equations at large accelerations.

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# 1 Introduction

One of the interesting aspects of the Dirac equation is the connection with the dynamics of a classical spinning particle. Since the Dirac equation describes a particle with spin, such a connection is expected in the classical limit of the Dirac theory. For a nonrelativistic particle in a weak electromagnetic field, this is a well-known subject [1,2]. The connection is usually realized by a succession of Foldy-Wouthuysen (FW) transformations, which projects the Hamiltonian on a Pauli two-spinor, from which the classical limit can be easily read. Some years ago, the result was extended also to relativistic particles in strong electromagnetic fields [3]. Using a more sophisticated FW transformation, it was shown there that the evolution equations for the Heisenberg operators in the FW representation with  $\hbar \rightarrow 0$  exactly reproduce the corresponding classical equations of motion for a particle with spin. In particular, the evolution of the quantum spin reduces to the classical evolution defined by the Bargman-Michel-Telegdi equation [4].

A similar situation appears to be valid for a test particle in the gravitational field. The generally accepted equations of motion in this case are the Mathisson-Papapetrou-Dixon<sup>1</sup> (MPD) equations [5–7]. The fact that the MPD equations can be extracted from the classical limit of the Dirac equation (neglecting terms quadratic in spin) has been established using various methods by many authors [9–17], with the observation that the connection is by no means unique. A main source of ambiguity for a particle in a curved background is the nonunique definition of the center of mass and implicitly of the spin of the particle, which is intimately linked to the freedom of choice of the supplemental condition which has to be imposed on the spin tensor in order to close the MPD equations [18]. It seems that there is yet no general agreement on which the “correct” supplemental conditions are, and whether different supplemental conditions lead or not to equivalent physical pictures [18,19], and the semiclassical limit of the Dirac equation might shed a light on these questions [16,17].

The intention of this paper is to investigate the quantum-classical connection in the Dirac theory, by considering a concrete problem in the usual flat spacetime. The problem is as follows: a particle bounces off from a perfectly reflecting plane. It is clear that the picture in the orbital space has nothing special, so that the interesting part is the evolution of the spin. Our aim is to determine the rotation of the spin after the reflection from the plane in the quantum and the classical description, and make a comparison between the two

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<sup>1</sup>For a history of the subject and recent developments see ref. [8].

results. We are especially interested in this problem due to the following reason. In the quantum description, we will consider a family of boundary conditions on the reflecting plane. In the classical description we will consider the simple case in which there is no specific interaction between the spin and the plane. It is then a good question which boundary conditions are the best fit for the classical result. Although this is a rather simple problem, it seems that it has not been investigated yet. We will see that some notable features emerge. We stress in advance that in both descriptions the solution can be exactly obtained, which will allow a perfectly accurate comparison between the quantum and the classical picture.

We begin with an outline of the calculation. In the quantum description, the key ingredient are the boundary conditions to be imposed on the perfectly reflecting plane. The common choices in literature for the Dirac field are the MIT [23–25] and spectral [20–22] boundary conditions. We will use the MIT boundary conditions, which are the closest ones to the usual Dirichlet or Neumann conditions for the scalar field. (In both cases the normal component of the density current on the boundary is required to vanish.) It will be also relevant for the comparison with the classical picture to consider the extension of the MIT conditions [26, 27] involving a chiral angle  $\phi$ . We recall that the effect of a nonvanishing chiral angle is to introduce an additional interaction between the field and the boundary, and that this extra interaction can significantly change the physics of the system. (A well-known example is the chiral bag models for nucleons [27–30]; see also below.)

Two important observations for our discussion are as follows. First, the nonchiral case  $\phi = 0$  corresponding to the original MIT boundary condition is the precise equivalent of an infinite repulsive scalar potential outside the region enclosed<sup>2</sup> by the reflecting boundary [23]. It is then immediate from the scalar nature of the potential that in this case there is no spin-dependent interaction between the particle and the boundary. By contrast, a nonvanishing angle  $\phi \neq 0$  generally introduces a specific interaction between the boundary and the spin [30]. A simple example which illustrates this fact is provided by the free Dirac particle in the exterior of a perfectly reflecting sphere with chiral MIT conditions on the surface [31]. It turns out for  $\phi = 0$  only scattering states exist, but for  $\phi \neq 0$  a finite number of bound states are also allowed. These bound states can be naturally interpreted in terms of an attraction force between the sphere and the particle. More specifically, with an appropriate chiral transformation one can show that a nonzero angle  $\phi \neq 0$  is equivalent to

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<sup>2</sup>Or, equivalently, an infinite mass of the field in this region.

the nonchiral MIT boundary condition, plus a delta-type magnetic field localized on the surface of the sphere. The attraction force between the particle and the chiral sphere can then be understood as a consequence of the interaction between the particle's spin (magnetic moment) and this effective magnetic field [31].

Returning to our goal, the calculation of the orientation of the spin in the reflected component of the wave is directly similar to that for the polarization of an electromagnetic plane wave scattered by a perfectly reflecting plane. The only essential difference is that we now have to use plane wave solutions of the Dirac equation. As expected, the MIT boundary conditions plus the kinematic constraints in the orbital space completely determine the relation between the incident and the reflected spin. This relation can be expressed in terms of a  $SU(2)$  rotation matrix, from which we will extract the rotation axis and angle of the spin.

We now turn to the classical problem. We will make the key assumption that the interaction between the particle and the reflective plane is such that no external torque acts on the particle. As a consequence, the spin four-vector is Fermi-Walker transported along the trajectory. We will consider the evolution of the three-spin in the proper frame of the particle, with the particle's proper frame defined via a Lorentz boost with respect to the laboratory frame. In these conditions, the Fermi-Walker transport implies that the spin evolves according to the Thomas precession formula [32, 33]. It is clear that on the inertial parts of the trajectories, *i.e.* before and after the impact with the plane, the spin remains fixed. The nontrivial part is what happens at the impact point. A first observation is that, assuming a perfectly reflecting plane, the velocity is non-differentiable at this point, which is rather unphysical. We will remediate this situation by considering that the reflecting plane acts via a repulsive potential, smoothing thus the trajectory. The evolution of the spin will then be determined by the Thomas precession due to the accelerated motion in this potential. Let us assume that the reflecting plane coincides with the  $xy$ -plane. We will naturally choose the repulsive potential to be of the form  $V(z)$ , assuring thus the translational invariance of the system along the plane. At first sight, one would expect the orientation of the spin after reflection to depend on the form of the potential  $V(z)$ . Notably, we will find that this is not so. Hence, the comparison with the quantum picture will not be affected by the ambiguity in the choice of the repulsive potential associated to the plane.

In brief, the main conclusions are as follows. Not surprisingly, we will find that the closest similarity between the quantum and the classical picture arises

only for a chiral angle  $\phi = 0$ .<sup>3</sup> For this particular value of  $\phi$  the rotation axes of the spin in the two cases coincide, and for nonrelativistic velocities of the incident particle the rotation angles also coincide. Moreover, the two rotation angles remain remarkably close up to velocities comparable with the speed of light. These show that for a vanishing chiral angle and not too high velocities the evolution of the spin at the impact with the plane is completely described by the Thomas precession effect. This is precisely what one would expect from the fact that the nonchiral MIT conditions correspond to a scalar potential associated to the surface, in which case no specific spin-dependent interaction exist.

By contrast, in the ultrarelativistic limit a rather unexpected discrepancy shows up, and this is perhaps the most interesting result in the paper: the rotation angle of the spin in the quantum picture remains finite, while in the classical picture the rotation angle diverges. The divergent angle in the classical problem seems to be unnatural, which will invite us to speculate that something goes wrong in the classical calculation. We will suggest that a more realistic model for a spinning particle which includes the elasticity of the body would leave the rotation angle in the ultrarelativistic limit finite. Generalizing, we will argue that a similar problem could appear for the MPD equations at large accelerations of the particle.

The paper is organized as follows. In sect. 2 we determine the rotation of the spin in the quantum problem. In sect. 3 we obtain the analogous quantity in the classical problem, and in sect. 4 we compare the two results. We end in sect. 5 by presenting the conclusions and the possible implications for the MPD equations. Throughout the paper we use natural units with  $\hbar = c = 1$ .

## 2 The quantum rotation angle

The first step is to obtain the wave function in the presence of the reflecting plane. As usual, the result can be written as a superposition of an incident and a reflected component. For a definite momentum of the particle, these components are plane waves solutions of the Dirac equation. It is useful to first recall the form of these solutions. We naturally restrict to the positive energy solutions.

We work in the standard Dirac representation in which the gamma matrices

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<sup>3</sup>As we will see there is actually a second possibility  $\phi = \pi$ , but this is equivalent to the case  $\phi = 0$ .

are ( $\sigma_i$  are the Pauli matrices)

$$\gamma^0 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}, \quad \gamma^i = \begin{vmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{vmatrix}, \quad i = 1, 2, 3. \quad (1)$$

The positive energy plane wave solutions can be written as (standard notation is used)

$$u_{p,\xi}(x) = u(p, \xi) e^{-ip \cdot x}, \quad (2)$$

where  $p^\mu = (E, \mathbf{p})$  is the four-momentum of the particle and  $\xi$  a unit normalized two-spinor which defines the spin. The four-spinors  $u(p, \xi)$  can be obtained as follows. One first introduces the spinors in the proper frame of the particle corresponding to the four-momentum  $\hat{p}^\mu = (m, \mathbf{0})$ , *i.e.*

$$u(\hat{p}, \xi) = \begin{vmatrix} \xi \\ 0 \end{vmatrix}, \quad \xi^\dagger \xi = 1. \quad (3)$$

The spin in the particle's proper frame is then defined by  $\xi$  the same as in the nonrelativistic theory. The four-spinors  $u(p, \xi)$  can be obtained as

$$u(p, \xi) = S(\Lambda_p) u(\hat{p}, \xi), \quad (4)$$

where  $\Lambda_p$  is a Lorentz transformation which sends  $\hat{p}$  into  $p$ . We choose it as usual to be a pure Lorentz boost,

$$S(\Lambda_p) = e^{-i\alpha \mathbf{n} \cdot \mathbf{K}}, \quad K_i = \frac{i}{2} \gamma_i \gamma_0, \quad (5)$$

where

$$\alpha = \operatorname{arctanh} \frac{|\mathbf{p}|}{E}, \quad \mathbf{n} = \frac{\mathbf{p}}{|\mathbf{p}|}. \quad (6)$$

The result is [1, 2]:

$$u(p, \xi) = \begin{vmatrix} \cosh \frac{\alpha}{2} \xi \\ \sinh \frac{\alpha}{2} \sigma_{\mathbf{n}} \xi \end{vmatrix}, \quad \sigma_{\mathbf{n}} \equiv \mathbf{n} \cdot \boldsymbol{\sigma}. \quad (7)$$

By fixing the Lorentz transformations  $\Lambda_p$  the proper frame of the particle is completely determined, making thus precise the significance of  $\xi$ . Note that the same proper frame of the particle will be used in the classical calculation, which will make immediate the comparison with the quantum result.

We now discuss the boundary conditions on the reflecting plane. Let us introduce the four-vector

$$N^\mu = (0, \mathbf{N}), \quad (8)$$

where  $\mathbf{N}$  is the unit normal to the plane. The MIT conditions with a chiral angle  $\phi$  are defined by [26, 27]

$$N^\mu \gamma_\mu \psi = i e^{i\phi \gamma^5} \psi, \quad (9)$$

with  $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ . We recall that in the right member

$$e^{i\phi \gamma^5} = \cos \phi I + i \sin \phi \gamma_5, \quad (10)$$

and that in the Dirac representation

$$\gamma^5 = \begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}. \quad (11)$$

In the case of interest, the reflecting surface is the  $xy$ -plane, and thus<sup>4</sup>

$$\mathbf{N} = \mathbf{e}_z. \quad (12)$$

Let us decompose the Dirac wave function as

$$\psi = \begin{vmatrix} \Phi \\ \chi \end{vmatrix}. \quad (13)$$

Combining the relations above one finds that condition (9) translates into

$$(\sigma_3 + \sin \phi) \Phi - i \cos \phi \chi = 0, \quad (14)$$

$$(\sigma_3 - \sin \phi) \chi + i \cos \phi \Phi = 0. \quad (15)$$

One can easily check that eqs. (14) and (15) are linearly dependent, as it should, in order to admit a non-zero solution. It is therefore sufficient to consider only one of the two equations.

Let us choose the directions of the incident ( $I$ ) and reflected ( $R$ ) directions as (see fig. 1)

$$\mathbf{n}_I = (\cos \theta, 0, -\sin \theta), \quad \mathbf{n}_R = (\cos \theta, 0, \sin \theta). \quad (16)$$

The corresponding momenta are

$$p_{I/R}^\mu = m (\cosh \alpha, \sinh \alpha \mathbf{n}_{I/R}). \quad (17)$$

The total wave function can then be written as

$$\psi(x) = u(p_I, \xi_I) e^{-ip_I \cdot x} + u(p_R, \xi_R) e^{-ip_R \cdot x}, \quad (18)$$

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<sup>4</sup>The sign ambiguity in the normal  $\mathbf{N}$  can be included in  $\phi \rightarrow \phi \pm \pi$ .

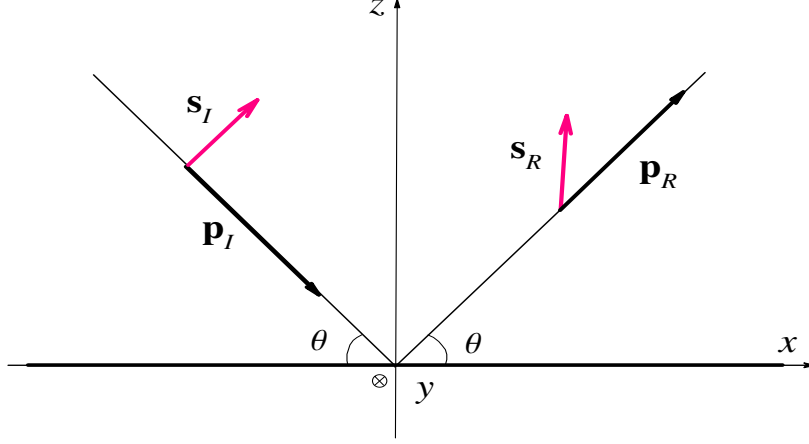


Figure 1: The geometry of the problem.

where the two four-spinors are

$$u(p_I, \xi_I) = \begin{vmatrix} \cosh \frac{\alpha}{2} \xi_I \\ \sinh \frac{\alpha}{2} \sigma_I \xi_I \end{vmatrix}, \quad \sigma_I = \mathbf{n}_I \cdot \boldsymbol{\sigma}, \quad (19)$$

$$u(p_R, \xi_R) = \begin{vmatrix} \cosh \frac{\alpha}{2} \xi_R \\ \sinh \frac{\alpha}{2} \sigma_R \xi_R \end{vmatrix}, \quad \sigma_R = \mathbf{n}_R \cdot \boldsymbol{\sigma}. \quad (20)$$

It is clear that the spinors  $\xi_I$  and  $\xi_R$  define the orientations of the incident and the reflected spin.

We are interested in the relation between  $\xi_I$  and  $\xi_R$ . Applying one of the conditions (14) or (15) to the wave function (18) on the plane  $z = 0$ , a simple calculation shows that the result can be put in the following form:

$$Q_I \xi_I = Q_R \xi_R, \quad (21)$$

where  $Q_I$  and  $Q_R$  are the matrices

$$Q_I = AI + \mathbf{B} \cdot \boldsymbol{\sigma}, \quad Q_R = A^* I + \mathbf{B} \cdot \boldsymbol{\sigma}, \quad (22)$$

with  $A$  and  $\mathbf{B}$  given by

$$A = 1 + i \cos \phi \sin \theta \tanh \frac{\alpha}{2}, \quad (23)$$

$$\mathbf{B} = \cos \phi \cos \theta \tanh \frac{\alpha}{2} \mathbf{e}_y + \sin \phi \mathbf{e}_z. \quad (24)$$



Note that  $A$  is complex and  $\mathbf{B}$  is real. Introducing the new matrix

$$\mathcal{U} = Q_R^{-1} Q_I, \quad (25)$$

eq. (21) becomes

$$\xi_R = \mathcal{U} \xi_I. \quad (26)$$

Using the obvious properties

$$Q_I^+ = Q_R, \quad [Q_I, Q_I^+] = 0, \quad (27)$$

it is easy to show that  $\mathcal{U}$  is a unitary matrix

$$\mathcal{U}^+ \mathcal{U} = I. \quad (28)$$

This means that the matrix  $\mathcal{U}$  can be readily interpreted as a rotation operator in the spin space.

Let us denote by  $\varphi_q$  and  $\mathbf{n}_q$  the rotation angle and, respectively, the (unit-norm) rotation axis implied by  $\mathcal{U}$ . We can write

$$\mathcal{U} = e^{i\chi} \left( \cos \frac{\varphi_q}{2} I - i \sin \frac{\varphi_q}{2} \mathbf{n}_q \cdot \boldsymbol{\sigma} \right), \quad (29)$$

where  $e^{i\chi}$  is a pure phase and where in the parentheses one recognizes the standard form of an SU(2) matrix in the angle-axis parameterization. Constructing the matrix  $\mathcal{U}$  from eqs. (21)-(25) and comparing with eq. (29), one finally finds:

$$\cos \frac{\varphi_q}{2} = \frac{|A|^2 - \mathbf{B}^2}{|A^2 - \mathbf{B}^2|}, \quad \sin \frac{\varphi_q}{2} = \frac{2 \operatorname{Im} A |\mathbf{B}|}{|A^2 - \mathbf{B}^2|}, \quad \mathbf{n}_q = \frac{\mathbf{B}}{|\mathbf{B}|}. \quad (30)$$

This practically solves the quantum problem. Note that  $\varphi_q$  and  $\mathbf{n}_q$  are completely determined by three parameters: the velocity parameter  $\alpha$ , the incident angle  $\theta$  and the chiral angle  $\phi$  (the mass  $m$  of the particle is irrelevant). The expressions (30) in the general case are somewhat complicated and we will not write them down in explicit form. A significant simplification will occur when we will make contact with the classical picture in sect. 4.

### 3 The classical rotation angle

We first recall some basic facts about the evolution of the classical spin. Let us denote the spin four-vector of the particle by  $S^\mu(\tau)$ , with  $\tau$  the proper time along the trajectory. Assuming that no external torque acts on the particle, the evolution of the spin is given by the Fermi-Walker transport

$$\frac{dS^\mu}{d\tau} = \Omega^\mu{}_\nu S^\nu, \quad \Omega^\mu{}_\nu = u^\mu a_\nu - a^\mu u_\nu, \quad (31)$$

where  $u^\mu$  and  $a^\nu$  are the four-velocity and four-acceleration of the particle. Using the Lorentz transformations  $\Lambda_p$  introduced in the previous section, the spin four-vector in the particle's proper frame is

$$\hat{S}^\mu(\tau) = [\Lambda_{p(\tau)}^{-1}]^\mu{}_\nu S^\nu(\tau), \quad p(\tau) = mu(\tau). \quad (32)$$

The orthogonality requirement between the four-spin and the four-velocity

$$S^\mu u_\mu = 0, \quad (33)$$

implies that  $\hat{S}^\mu$  is of the form

$$\hat{S}^\mu = (0, \mathbf{s}), \quad (34)$$

where the three-vector  $\mathbf{s}$  can be identified with the usual spin measured in the particle's proper frame. It is clear that with these definitions  $\mathbf{s}$  is the direct correspondent of the spin determined by the spinor  $\xi$  in the quantum problem.

As is well known, the evolution of the spin  $\mathbf{s}$  is defined by the Thomas precession [32, 33].

$$\frac{d\mathbf{s}}{dt} = \boldsymbol{\omega}_T \times \mathbf{s}, \quad \boldsymbol{\omega}_T = \frac{\gamma^2}{\gamma + 1} \mathbf{a} \times \mathbf{v}, \quad (35)$$

where  $\mathbf{v}$  and  $\mathbf{a}$  are the classical velocity and acceleration of the particle, and  $\gamma$  is the usual relativistic factor. We now apply eqs. (35) to determine the relation between the incident and the reflected spin in the classical problem.

On the inertial parts of the trajectory when the particle is not in contact with the plane the acceleration is  $\mathbf{a} = 0$ , which implies  $\boldsymbol{\omega}_T = 0$ , and thus  $\mathbf{s} = \text{constant}$ . The nontrivial evolution of the spin is determined by the interaction with the plane. As discussed in sect. 1, the idea is to describe this interaction via a region of repulsive potential  $V(z)$ , where we will choose the potential to be zero for  $z > 0$ . The picture in this case looks like in fig. 2. We are interested in the rotation of the spin after passing through the repulsive region  $z \leq 0$ . We will make the simplifactory assumption that the spin-orbit coupling  $E_{s-o} = \mathbf{s} \cdot \boldsymbol{\omega}_T$  is sufficiently small compared with the potential energy  $E_{\text{pot}} = V(z)$ , so that the trajectory in the orbital space is practically the same with that of a particle without spin. This would correspond to a sufficiently small spin  $\mathbf{s}$  and/or a sufficiently large mass  $m$  of the particle, which will keep the acceleration  $\mathbf{a}$  small, and thus the precession  $\boldsymbol{\omega}_T$  small.

It is clear from the symmetries of the problem that we can consider that the trajectory is contained in the  $xz$ -plane. Let us denote the by  $\psi$  the angle

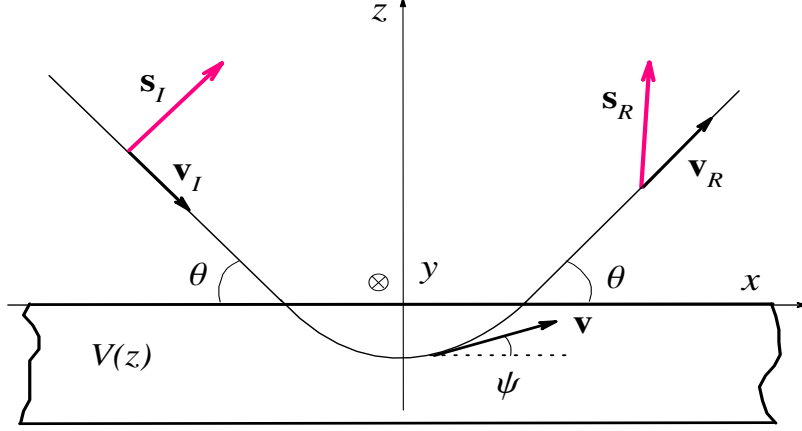


Figure 2: The classical trajectory in the presence of the reflecting surface modeled by a repulsive potential in the region  $z \leq 0$ .

between the velocity  $\mathbf{v}$  and the  $x$ -axis. One can read from the fig. 2 that for the piece of the trajectory  $z \leq 0$  this angle varies within the interval

$$\psi \in [-\theta, \theta]. \quad (36)$$

The trick is to parameterize the velocity in the following form:

$$\mathbf{v}(\psi) = v(\psi) \mathbf{e}(\psi), \quad \mathbf{e}(\psi) = (\cos \psi, 0, \sin \psi), \quad (37)$$

where  $\psi$  has to be seen as a function of the laboratory time  $t$ . The acceleration of the particle then is

$$\mathbf{a}(\psi) = \frac{d\mathbf{v}}{dt} = \left( \frac{dv}{d\psi} \mathbf{e}(\psi) + v(\psi) \frac{d\mathbf{e}}{d\psi} \right) \frac{d\psi}{dt}. \quad (38)$$

Introducing eqs. (37) and (38) in the Thomas precession formula (35), one finds that the precession vector is (in obvious notation)

$$\boldsymbol{\omega}_T = \frac{d\varphi_c}{dt} \mathbf{n}_c, \quad (39)$$

where the angular velocity and the rotation axis are

$$\frac{d\varphi_c}{dt} = \frac{\gamma^2 v^2}{\gamma + 1} \frac{d\psi}{dt}, \quad \mathbf{n}_c = \mathbf{e}_y. \quad (40)$$

Note that, essentially, the rotation axis is independent of time. In these conditions the total rotation angle of the spin  $\varphi_c$  can be simply obtained by integrating the angular velocity  $d\varphi_c/dt$  with respect to  $t$ . Due to the derivative factor  $d\psi/dt$  in the angular velocity, the integration can be readily replaced by that with respect to  $\psi$  with the integration limits (36). One thus obtains

$$\varphi_c = \int_{-\theta}^{\theta} d\psi \frac{\gamma^2 v^2}{\gamma + 1}, \quad (41)$$

where it remains to find the dependence  $v = v(\psi)$ . This can be easily done by noting that the translational invariance along the plane ensures the conservation of the momentum along the axis  $x$ , which implies

$$\gamma(\psi)v(\psi)\cos\psi = \gamma v \cos\theta, \quad (42)$$

where in the left member are the quantities before/after on the impact on the plane. Using eqs. (41) and (42) a few manipulations lead to

$$\varphi_c = (\gamma v \cos\theta)^2 \times \int_{-\theta}^{\theta} \frac{d\psi}{\cos^2\psi} \left( \sqrt{1 + \frac{(\gamma v \cos\theta)^2}{\cos^2\psi}} + 1 \right)^{-1}. \quad (43)$$

This solves the classical problem. Thus, the picture is that the spin rotates around the axis  $\mathbf{e}_y$  with the angle  $\varphi_c$  defined by eq. (43). As anticipated, the rotation angle is independent of the form of the potential  $V(z)$ . We will explore some consequences of our results in the following section.

## 4 Comparison between the two results

We are interested in the values of the chiral angle  $\phi$  which lead to the closest similarity between the evolution of the spin in the quantum and the classical picture. We begin by looking at the rotation axes. From eqs. (30) and (24) the axis in the quantum problem is

$$\mathbf{n}_q \sim \cos\phi \cos\theta \tanh\frac{\alpha}{2} \mathbf{e}_y + \sin\phi \mathbf{e}_z. \quad (44)$$

We have seen that in the classical problem  $\mathbf{n}_c = \mathbf{e}_y$ . Comparing with eq. (44) it is immediate that the two axes coincide only when  $\sin\phi = 0$ , *i.e.*

$$\phi = 0 \text{ or } \pi. \quad (45)$$

A simple calculation using eqs. (23), (24) and (30) shows that the case  $\phi = 0$  implies

$$\tan\frac{\varphi_q}{2} = \frac{\sin 2\theta \tanh^2(\alpha/2)}{1 - \cos 2\theta \tanh^2(\alpha/2)}, \quad \mathbf{n}_q = \mathbf{e}_y. \quad (46)$$

Using the same formulas one finds that the case  $\phi = \pi$  is equivalent to replacing in the expressions above

$$\varphi_q \rightarrow -\varphi_q, \quad \mathbf{n}_q \rightarrow -\mathbf{n}_q. \quad (47)$$

This brings nothing new, as it defines the same rotation<sup>5</sup> as eq. (46). In the rest of the section, we will therefore continue to refer only to the case  $\phi = 0$ .

We now compare the rotation angles. We begin by considering the nonrelativistic limit  $v \ll 1$ . In this limit we can approximate in eq. (46)

$$\alpha \simeq v, \quad \tanh(\alpha/2) \simeq v/2, \quad (48)$$

and since  $\tanh^2(\alpha/2) \ll 1$  the denominator of the long fraction can be approximated to unity. In these conditions the quantum angle becomes

$$\varphi_q^{NR} \simeq \frac{1}{2} v^2 \sin 2\theta. \quad (49)$$

In the classical angle (43) the relativistic factor is  $\gamma \simeq 1$ , and using  $v^2 \ll 1$  the inverse round brackets under the integral can be approximated to  $1/2$ . This leads to

$$\begin{aligned} \Delta\phi_c^{NR} &\simeq (v \cos \theta)^2 \times \int_{-\theta}^{\theta} \frac{d\psi}{2 \cos^2 \psi}, \\ &= \frac{1}{2} v^2 \sin 2\theta. \end{aligned} \quad (50)$$

The conclusion from eqs. (49) and (50) is that in the nonrelativistic limit the quantum and the classical rotation angles coincide.

It is interesting to see what happens at larger velocities  $v$ . In fig. 3 we represented the rotation angles as a function of  $\theta$  for different values of  $v$ . The curves show that the two angles are practically indistinguishable for velocities up to  $v \sim 0.1$ . Remarkably, the two angles remain fairly equal also for relativistic velocities as high as  $v \sim 1/2$ .

However, as  $v$  increases the classical angle tends to assume larger values and the curves become significantly different. A plot for a highly relativistic velocity  $v = 0.9$  is shown in fig. 4.

We now consider the ultrarelativistic limit  $v \rightarrow 1$ . In this limit in the quantum angle (46)  $\tanh \alpha/2 \rightarrow 1$ , from which

$$\tan \frac{\phi_q^{UR}}{2} \simeq \frac{\sin 2\theta}{1 - \cos 2\theta}, \quad (51)$$

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<sup>5</sup>This means that for  $\phi = 0$  the sign ambiguity in the normal  $\mathbf{N}$  in the boundary condition (9) is irrelevant. The same property appears in other results [34, 35]. The property however is not universal, as illustrated by the Casimir energies between two plates [36] or inside a cylinder [37] with MIT chiral boundary conditions.

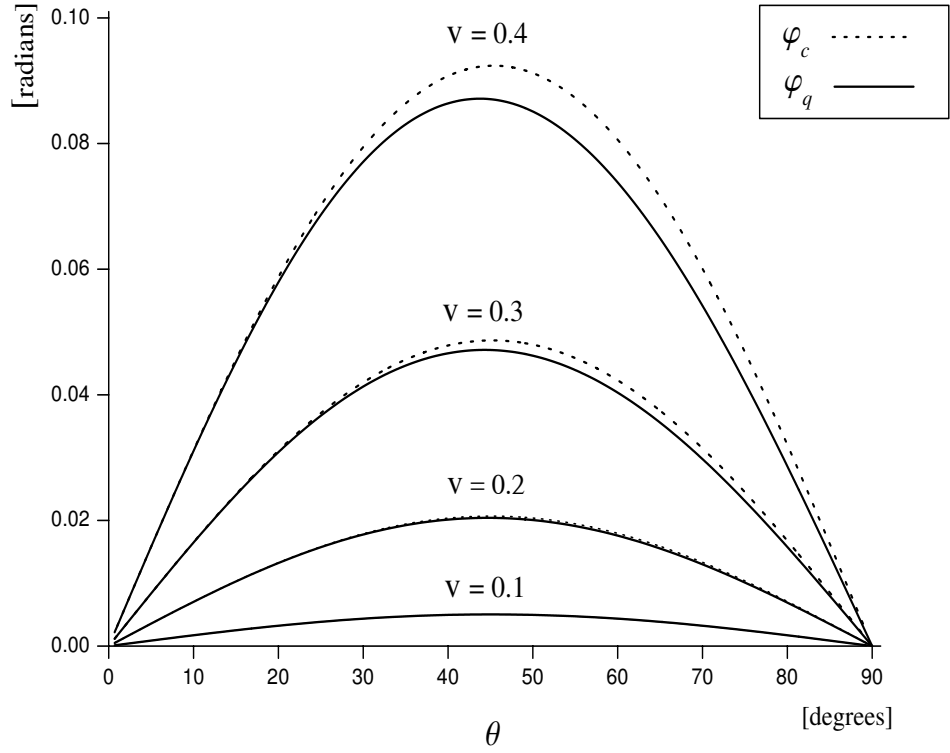


Figure 3: The rotation angles of the spin in the quantum (solid line) and the classical problem (dotted line) shown as a function of the angle  $\theta$  for a number of velocities  $v$  not too close to the speed of light  $c = 1$ . For velocities smaller than  $v \sim 0.1$  the curves are indistinguishable on the plot.

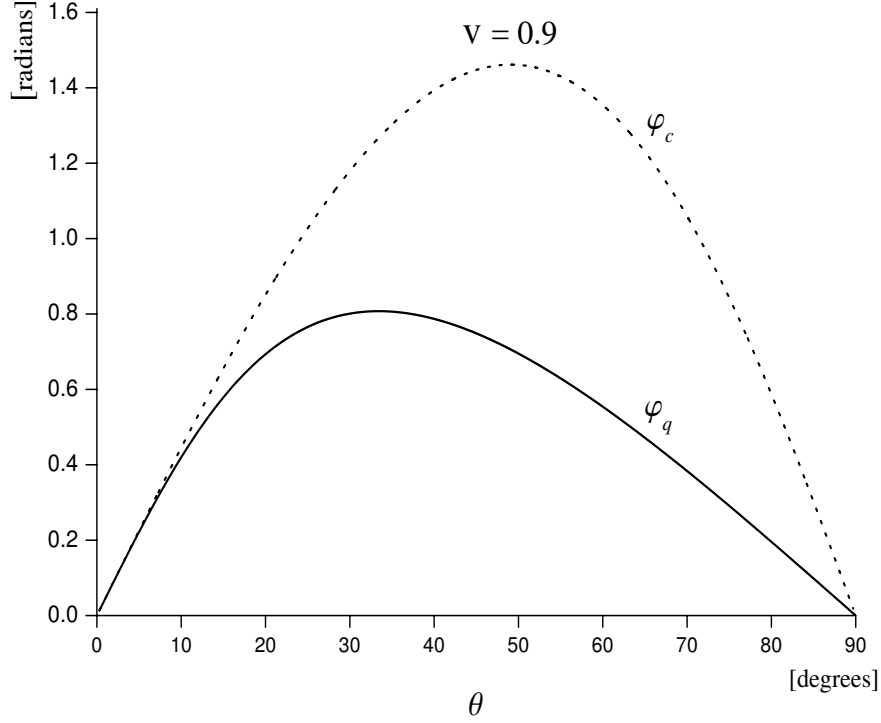


Figure 4: The same as in fig. 1 but for  $v = 0.9$ . For highly relativistic velocities the classical rotation angles generally become significantly larger than the quantum ones.

which is equivalent to

$$\varphi_q^{UR} \simeq \pi - 2\theta. \quad (52)$$

Note that the result remains finite. We will give a simple geometric interpretation for eq. (52) a few lines below. A plot which shows  $\varphi_q$  as a function of the velocity parameter  $\alpha$  for various angles  $\theta$  is given fig. 5.

A significantly different result is obtained in the classical picture. In the ultrarelativistic limit  $\gamma \rightarrow \infty$ , and counting the powers of  $\gamma$  in eq. (43) one can see that that  $\varphi_c$  behaves as

$$\varphi_c^{UR} \sim \gamma \rightarrow \infty. \quad (53)$$

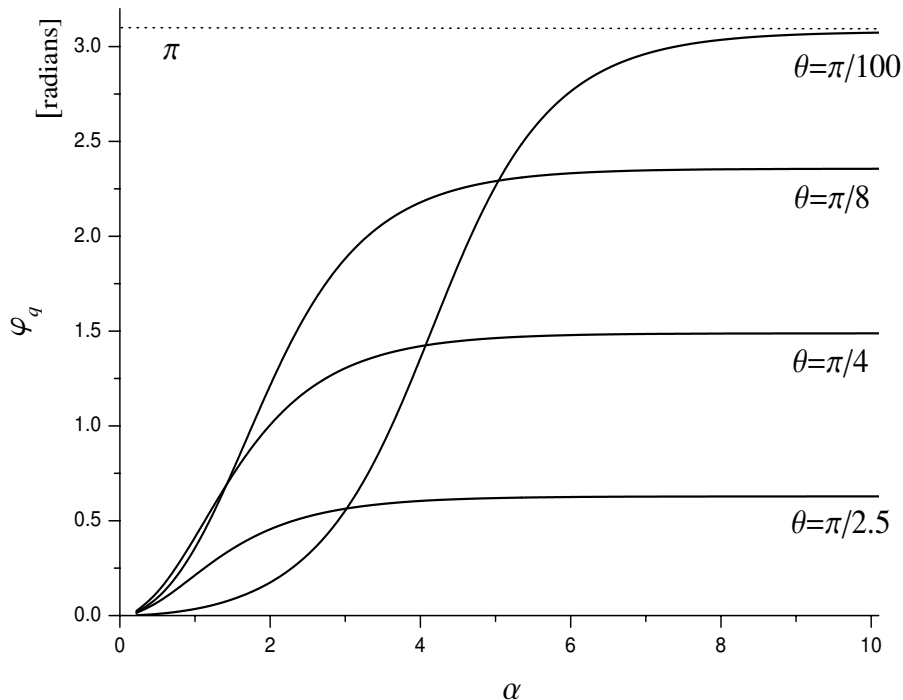


Figure 5: The quantum rotation angle  $\varphi_q$  as a function of the velocity parameter  $\alpha$  for different angles  $\theta$ . The horizontal pieces of the curves at large  $\alpha$  correspond to the ultrarelativistic limit (52).

This means that in the ultrarelativistic limit the rotation angle diverges (*i.e.*, the spin makes an infinite number of turns). A plot which illustrates the divergent behavior of  $\varphi_c$  compared to that of  $\varphi_q$  is shown in Figs. 6. We will comment on the discrepancy between the quantum and the classical result in the next section.

Finally, although not directly related to our subject, it deserves to observe that eq. (52) admits the following interpretation. Note first that, after the reflection from the plane, the momentum  $\mathbf{p}$  of the particle is rotated around the axis  $\mathbf{e}_y$  (*i.e.* the same as for the spin), and the rotation angle is equal to  $-\theta < 0$  (the rotation is in the counterclockwise direction and hence the negative



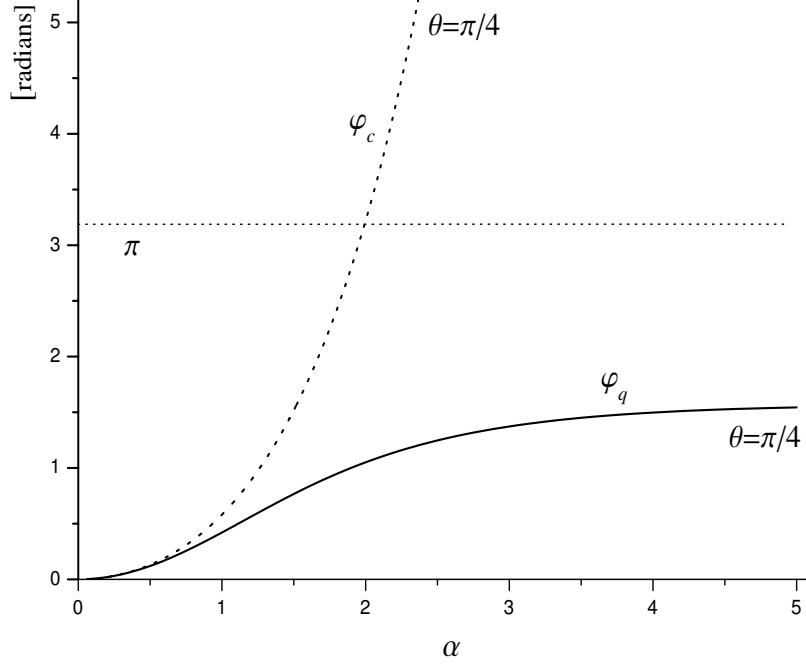


Figure 6: The quantum and the classical rotation angles as functions of  $\alpha$  for a fixed incidence angle  $\theta$ . In the ultrarelativistic limit  $\varphi_q$  remains finite, while  $\varphi_c$  diverges.

sign). It follows that the rotation of the spin  $\mathbf{s}$  relative to the direction of  $\mathbf{p}$  is

$$\varphi_c^{UR} \simeq \pi - 2\theta - (-2\theta) = \pi. \quad (54)$$

It is easy to see that this implies that the projection of  $\mathbf{s}$  onto the direction of  $\mathbf{p}$  changes sign after the impact with the plane. In other words, the helicity of the particle changes sign after the impact. At first sight, this might appear counterintuitive if one takes the view that in the ultrarelativistic limit the mass of the particle can be ignored, and thus one can assume that we deal with massless fermions. In this case one might think that we could restrict to definite helicity states  $h = \pm 1/2$ , which would be in contradiction with (54). The argument however is not valid, if one recalls that for massless fermions such states are equivalent to definite chirality states. The point is that the MIT boundary

conditions (9) are not chiral invariant [23], so that they are incompatible with a dynamics involving only well-defined chiral states. The helicity flip (54) has thus to be seen intimately connected with the massive nature of the field.

## 5 Conclusions and discussions

We considered a Dirac particle which bounces off from a perfectly plane described by chiral MIT boundary conditions and determined the rotation of the spin after the reflection from the plane. We made a comparison with the analogous result in the classical version of the problem, in which case the reflecting plane was described by a repulsive scalar potential with translational invariance along the plane. In the classical problem we also assumed that no external torque acts on the particle, so that the evolution of the spin is completely determined by the Thomas precession effect. Under these conditions, it turned out the rotation of the classical spin is independent on the form of the repulsive potential.

We have found that (1) the rotation axes of the spin in the two descriptions coincide only for the chiral angle  $\phi = 0$  (or, equivalently,  $\phi = \pi$ ), (2) for these values of the chiral angle the rotation angles coincide in the nonrelativistic limit, and (3) the two rotation angles show a remarkably good agreement up velocities comparable to the speed of light. By contrast, (4) in the ultrarelativistic limit the quantum rotation angle remains finite, while the classical angle diverges.

Points (1)-(3) are consistent with the fact that a chiral angle  $\phi = 0$  introduces no specific interaction between the spin and the reflecting plane [23], which at classical level means that the spin evolves as we assumed in the classical calculation. This supports the general picture that the dynamics of the spin at the impact with a reflecting surface with  $\phi = 0$  is completely contained in the Thomas precession effect, and thus is of purely mechanical nature. However, our result makes it clear that a complete quantum-classical similarity arises only in the nonrelativistic limit. At sufficiently high velocities quantum effects become important and the description based on the Thomas precession becomes inadequate.

Points (1) and (2) are somehow similar to the recovery of the nonrelativistic spin-orbit coupling from the Dirac equation in an external electromagnetic field, in which case the result is given by the Thomas term plus a magnetic interaction term, where the last one arises due to the coupling between the spin and the magnetic field in the proper frame of the particle. In our case no such coupling exists, and thus only the Thomas term is determinant. It might be perhaps

interesting to see what type of interaction between the spin and the plane would assure the correspondence with the quantum picture for  $\phi \neq 0$ . For example, a possible solution could be as in ref. [31] a delta-type magnetic field localized on the plane.

The rather unexpected result of our calculation is point (4). The infinite difference between the two rotation angles implies a serious discrepancy between the quantum and the classical picture in the high velocity limit. In the rest of the section, we will make a few observations on this fact.

One can adopt essentially two views at this point. One is to simply admit that there is no reason for the spin of a classical particle to reproduce the behavior of the spin of a quantum one, and thus there is no inconsistency between the two results. This would then be just another instance in which the quantum and the classical theory lead to qualitatively different pictures (*e.g.*, one can think of zero-point energy of a quantum and a classical oscillator).

Another option would be to consider that the divergent angle in the classical picture is unphysical, and thus something goes wrong in the classical calculation (*e.g.*, one can think of the ultraviolet catastrophe in the blackbody radiation problem). We incline to believe that this is the case. In the following, we will comment on this possibility.

A frequent cause of an unphysical result in a theoretical model is an unrealistic idealization in the model. In the case considered here, such an idealization can be identified in the fact that the particle is assumed to behave as a perfectly rigid body. If one recalls the derivation of the Thomas precession formula (35), this is implicit in the fact that the body is assumed to rigidly rotate as a whole together with the Fermi-Walker transported axis of the particle's proper frame [32].

A more realistic description would be to treat the classical particle as a finite size elastic body. It is then tempting to conjecture that for such particles the rotation angle of the spin will stay finite in the ultrarelativistic limit. Notice that elasticity effects will become important precisely at large velocities of the particle. Such velocities would imply large accelerations during the impact with the plane, which in turn can lead to large deformations of the body. To prove this conjecture however is beyond the scope of this paper, and most probably is not a simple task. Unfortunately, in order to solve this problem one should have at hand the relativistic equations of motion for a deformable spinning body, which seems to be an unclarified issue yet.<sup>6</sup>

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<sup>6</sup>To our knowledge, the most suitable models that could be applied to investigate our problem are the quasi-rigid bodies [38, 39] in which the MPD equations are supplemented by

We feel that an analogy might be relevant here. A long studied problem in classical electrodynamics is that of the dynamics of charges in interaction with their proper field. It is well-known that the equation of motion for pointlike charges (the Abraham-Lorentz-Dirac equation) admits unphysical solutions, *i.e.* acausal and self-accelerated trajectories [33, 45]. It is also well-known that the unphysical solutions can be eliminated if one considers extended charges with a sufficiently large radius [46–52]. Notably, the unphysical solutions can be also eliminated in the quantum theory,<sup>7</sup> in which case the associated Compton wave length acts as a sort of radius of the charge [53]. It is appealing to see the divergent rotation angle of the spin obtained here as an analogue of the unphysical trajectories for pointlike charges, with the perfect rigidity of the body corresponding to the idealized limit of a zero radius of the charge. The same as retardation effects in extended charges can eliminate the unphysical trajectories, one can hope that elasticity effects will keep the rotation angle of the spin finite in the ultrarelativistic limit. It is also perhaps not a coincidence that the divergent rotation angle arises in the high energy limit, which in the electrodynamic analogy would correspond to the unphysical divergent self-energy of pointlike charges.

Finally, a practical implication related to the above point could be the following. An astrophysical problem that has received an increased interest in the last years is that of the trajectories of spinning bodies in the vicinity of black holes. When the mass of the black hole is much larger than that of the orbiting body and the radius of the body is sufficiently small compared to the local radius of curvature of the spacetime, the motion is well described by the MPD equations [54, 55]. We recall that the MPD equations are the pole-dipole approximation in the multipole expansion of the Dixon’s equations, and that this approximation is justified for a sufficiently small body, so that the internal tidal forces can be ignored [7]. Practically, this means that the MPD equations can be applied only to negligibly small, perfectly rigid bodies. On the other hand, it is clear that for a realistic body in sufficiently strong gravitational fields the tidal forces can significantly deform the body, in which case the MPD will no longer apply.

Our observation is that such a situation would be similar to that in the quadrupole terms induced by the spin and the velocity of the body. Using these models it was shown that quadrupole effects at large accelerations can significantly alter the trajectory of the body [40–44]. See also the references in ref. [39], Sec. IV.

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<sup>7</sup>In the first quantized sense, as in our discussion. The solutions in question are the solutions of the Heisenberg equation of motion for the position operator of the charge [53] (in the nonrelativistic limit).

classical problem discussed here in the ultrarelativistic limit, in which case the large accelerations during the impact with the reflecting plane would be the analogue of the large accelerations in a strong gravitational field. The problematic divergent precession of the spin in our calculation can then be suspected to occur also in the solutions of the MPD equations in strong fields. One could further speculate from here that at very large velocities/accelerations the MPD equations will generally *overestimate* the precession of the spin compared to that of a finite size elastic object. In other words, elasticity effects will tend to decrease the precession of the spin. It might be of interest to see whether this phenomenon indeed occurs. For example, the effect might show up by comparing the predictions of the MPD equations with numerical simulations of the evolution of physically realistic relativistic bodies in strong gravitational fields.<sup>8</sup> In a more unsophisticated setting, one could consider the models of the quasi-rigid bodies mentioned above (see footnote 6) and examine the quadrupole effects in the precession of the spin for accelerated bodies. Our conjecture then would be that at large velocities/accelerations the quadrupole effects associated to the elasticity of the body will tend to decrease the precession compared to that predicted by the standard Thomas precession formula (35).

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<sup>8</sup>For example, there exists a wealth of work on numerical simulations for spinning coalescing neutron stars or spinning neutron stars spiralling into a black hole, see *e.g.* refs. [56–60].

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